Perturbation of Super-Sech Solitons in Dispersion-Managed Optical Fibers

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Abstract The dynamics of super-sech solitons in dispersion-managed optical fibers is obtained in this paper. The dynamical system of soliton parameters is obtained for such pulses for dispersion-managed fibers, in presence of various perturbation terms. The perturbation terms studied are Hamiltonian, as well as non-Hamiltonian along with non-local types.

1 Introduction

The propagation of solitons through optical fibers has been a major area of research given its potential applicability in all optical communication systems [1–10]. The field of telecommunications has undergone a substantial evolution in the last couple of decades due to the impressive progress in the development of optical fibers, optical amplifiers as well as transmitters and receivers. In a modern optical communication system, the transmission link is composed of optical fibers and amplifiers that replace the electrical regenerators. But the amplifiers introduce some noise and signal distortion that limit the system capacity. Presently the optical systems that show the best characteristics in terms of simplicity, cost and robustness against the degrading effects of a link are those based on intensity modulation with

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direct detection (IM-DD). Conventional IM-DD systems are based on non-return-to-zero (NRZ) format, but for transmission at higher data rate the return-to-zero (RZ) format is preferred. When the data rate is quite high, soliton transmission can be used. It allows the exploitation of the fiber capacity much more, but the NRZ signals offer very high potential especially in terms of simplicity [9].

There are limitations, however, on the performance of optical system due to several effects that are present in optical fibers and amplifiers. Signal propagation through optical fibers can be affected by group velocity dispersion (GVD), polarization mode dispersion (PMD) and the nonlinear effects. The chromatic dispersion that is essentially the GVD when waveguide dispersion is negligible, is a linear effect that introduces pulse broadening generates intersymbol interference. The PMD arises due the fact that optical fibers for telecommunications have two polarization modes, in spite of the fact that they are called monomode fibers. These modes have two different group velocities that induce pulse broadening depending on the input signal state of polarization. The transmission impairment due to PMD looks similar to that of the GVD. However, PMD is a random process as compared to the GVD that is a deterministic process. So PMD cannot be controlled at the receiver. Newly installed optical fibers have quite low values of PMD that is about $0.1 \text{ ps}/\sqrt{\text{km}}$.

The main nonlinear effects that arises in monomode fibers are the Brillouin scattering, Raman scattering and the Kerr effect. Brillouin is a backward scattering that arises from acoustic waves and can generate forward noise at the receiver. Raman scattering is a forward scattering from silica molecules. The Raman gain response is characterized by low gain and wide bandwidth namely about 5 THz. The Raman threshold in conventional fibers is of the order of 500 mW for copolarized pump and Stokes' wave (that is about 1 W for random polarization), thus making Raman effect negligible for a single channel signal. However, it becomes important for multichannel wavelength-division-multiplexed (WDM) signal due to an extremely wide band of wide gain curve.

The Kerr effect of nonlinearity is due to the dependence of the fiber refractive index on the field intensity. This effect mainly manifests as a new frequency when an optical signal propagates through a fiber. In a single channel the Kerr effect induces a spectral broadening and the phase of the signal is modulated according to its power profile. This effect is called self-phase modulation (SPM). The SPM-induced chirp combines with the linear chirp generated by the chromatic dispersion. If the fiber dispersion coefficient is positive namely in the normal dispersion regime, linear and nonlinear chirps have the same sign while in the anomalous dispersion regime they are of opposite signs. In the former case, pulse broadening is enhanced by SPM while in the later case it is reduced. In the anomalous dispersion case the Kerr nonlinearity induces a chirp that can compensate the degradation induced by GVD. Such a compensation is total if soliton signals are used.

If multichannel WDM signals are considered, the Kerr effect can be more degrading since it induces nonlinear cross-talk among the channels that is known as the cross-phase modulation (XPM). In addition WDM generates new frequencies called the Four-Wave mixing (FWM). The other issue in the WDM system is the collision-induced timing jitter that is introduced due to the collision of solitons in different channels. The XPM causes further nonlinear chirp that interacts with the fiber GVD as in the case of SPM. The FWM is a parametric interaction among waves satisfying a particular relationship called phase-matching that lead to power transfer among different channels.

To limit the FWM effect in a WDM it is preferable to operate with a local high GVD that is periodically compensated by devices having an opposite sign of GVD. One such device is a simple optical fiber with opportune GVD and the method is commonly known as the dispersion-management. With this approach the accumulated GVD can be very low and at

the same time FWM effect is strongly limited. Through dispersion-management it is possible to achieve highest capacity for both RZ as well as NRZ signals. In that case the overall link dispersion has to be kept very close to zero, while a small amount of chromatic anomalous dispersion is useful for the efficient propagation of a soliton signal. It has been demonstrated that with soliton signals, the dispersion-management is very useful since it reduces collision induced timing jitter [3] and also the pulse interactions. It thus permits the achievement of higher capacities as compared to the link having constant chromatic dispersion.

2 Governing Equations

The relevant equation, for studying the propagation of solitons through polarization-preserving optical fibers, is the nonlinear Schrödinger's equation (NLSE) with damping and periodic amplification [1]. In the dimensionless form, the NLSE is given by

$$iu_z + \frac{D(z)}{2}u_{tt} + |u|^2 u = -i\Gamma u + i\left[e^{\Gamma z_a} - 1\right]\sum_{n=1}^{N}\delta(z - nz_a)u \qquad (1)$$

Here, Γ is the normalized loss coefficient, z_a is the normalized characteristic amplifier spacing and z and t represent the normalized propagation distance and the normalized time, respectively, while u represents the wave profile expressed in the nondimensional units.

Also, $D(z)$ is used to model strong dispersion management. The fiber dispersion $D(z)$ is decomposed into two components namely a path-averaged constant value δ_a and a term representing the large rapid variation due to large local values of the dispersion [2]. Thus,

$$D(z) = \delta_a + \frac{1}{z_a}\Delta(\zeta) \qquad (2)$$

where $\zeta = z/z_a$. The function $\Delta(\zeta)$ is taken to have average zero over an amplification period namely

$$\langle\Delta\rangle = \frac{1}{z_a}\int_0^{z_a}\Delta\left(\frac{z}{z_a}\right)dz = 0 \qquad (3)$$

so that the path-averaged dispersion D will have an average δ_a namely

$$\langle D\rangle = \frac{1}{z_a}\int_0^{z_a}D(z)dz = \delta_a \qquad (4)$$

The proportionality factor in front of $\Delta(\zeta)$, in (2), is chosen so that both δ_a and $\Delta(\zeta)$ are quantities of order one. In practical situations, dispersion management is often performed by concatenating together two or more sections of given length with different values of fiber dispersion. In the special case of a two-step map it is convenient to write the dispersion map as a periodic extension of [2]

$$\Delta(\zeta) = \begin{cases} \Delta_1, & 0 \le |\zeta| < \frac{\theta}{2}, \\ \Delta_2, & \frac{\theta}{2} \le |\zeta| < \frac{1}{2} \end{cases} \qquad (5)$$

where Δ_1 and Δ_2 are given by

$$\Delta_1 = \frac{2s}{\theta} \qquad (6)$$
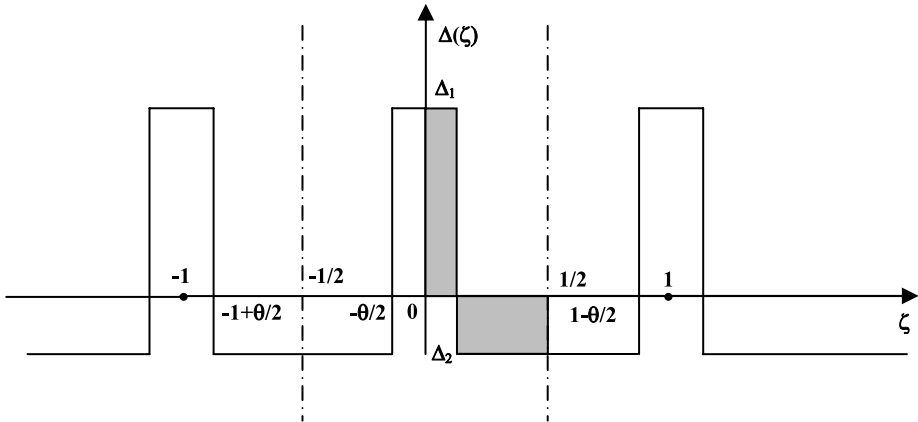
Fig. 1 Schematic diagram of a dispersion map

and

$$\Delta_2 = -\frac{2s}{1 - \theta} \tag{7}$$

with the map strength s defined as

$$s = \frac{\theta \Delta_1 - (1 - \theta)\Delta_2}{4} \tag{8}$$

Conversely,

$$s = \frac{\Delta_1 \Delta_2}{4(\Delta_2 - \Delta_1)} \tag{9}$$

and

$$\theta = \frac{\Delta_2}{\Delta_2 - \Delta_1} \tag{10}$$

A typical two-step dispersion map is shown in Fig. 1.

The solution to (1) is chosen of the form $u(z, t) = P(z)q(z, t)$, for real P. Taking P to satisfy

$$P_z + \Gamma P - \left[e^{\Gamma z_a} - 1\right] \sum_{n=1}^{N} \delta(z - nz_a)P = 0 \tag{11}$$

(1) transforms to

$$iq_z + \frac{D(z)}{2}q_{tt} + g(z)|q|^2 q = 0 \tag{12}$$

where

$$g(z) = P^2(z) = a_0^2 e^{-2\Gamma(z - nz_a)} \tag{13}$$

for $z \in [nz_a, (n + 1)z_a)$ and $n > 0$ and also

$$a_0 = \left[\frac{2\Gamma z_a}{1 - e^{-2\Gamma z_a}}\right]^{\frac{1}{2}} \tag{14}$$

so that over each amplification period

$$\langle g(z) \rangle = \frac{1}{z_a} \int_0^{z_a} g(z) dz = 1 \tag{15}$$

Equation (12) is commonly known as the Dispersion-Managed Nonlinear Schrödinger's equation (DM-NLSE) and it governs the propagation of a dispersion-managed soliton through an optical fiber with damping and periodic amplification.

## 3 Polarization Preserving Fibers

In a polarization preserved optical fiber, the propagation of solitons is governed by DM-NLSE given by (12). If $D(z) = g(z) = 1$ in (12), the NLSE is recovered. It is possible to integrate NLSE by the method of Inverse Scattering Transform (IST) and thus the NLSE falls into the category of S-integrable partial differential equations (15). The IST is the non-linear analog of Fourier transform that is used for solving linear partial differential equations. Moreover, the NLSE has an infinite number of conserved quantities. However, (12), as it appears, is no longer integrable and thus it does not belong to IST picture. Now, it is assumed that the solution of (12) is given by a super-sech (SS) chirped pulse of the form [2, 3, 6]

$$q(z, t) = \frac{A(z)}{\cosh^n [B(z)(t - \bar{t})]} \exp \left[iC(z) \left\{ t - \bar{t}(z) \right\}^2 - i\kappa(z) \left\{ t - \bar{t}(z) \right\} + i\theta(z) \right] \tag{16}$$

where the parameters $A(z)$, $B(z)$, $C(z)$, $\kappa(z)$, $\bar{t}(z)$ and $\theta(z)$ respectively represent the soliton amplitude, the inverse width of the pulse, chirp, frequency, the center of the pulse and the phase of the pulse. Also, the parameter $n > 0$. The SS pulses for various values of the parameter n is given in the following Fig. 2.

3.1 Integrals of Motion

Equation (12) does not contain an infinite number of integrals of motion as in the case of NLSE that gives classical solitons. In fact, there are as few as two of them. They are
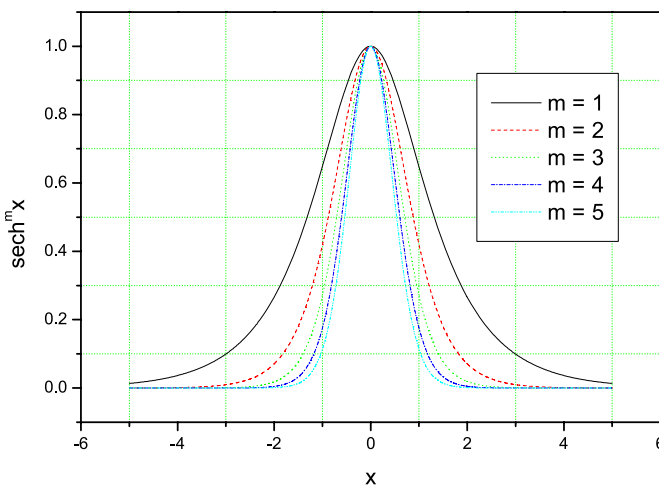
Fig. 2 Super-sech pulses

energy (E), also known as the L_2 norm, and linear momentum (M) that are respectively given by

$$E = \int_{-\infty}^{\infty} |q|^2 dt = \frac{A^2}{B} B\left(n, \frac{1}{2}\right) \tag{17}$$

$$M = \frac{i}{2} D(z) \int_{-\infty}^{\infty} (q^* q_t - q q_t^*) \, dt = -\kappa D(z) \frac{A^2}{B} B\left(n, \frac{1}{2}\right) \tag{18}$$

where in (17) and (18), $B(l, m)$ stands for the beta function. These integrals are evaluated by using the chirped SS soliton whose form is given in (16).

3.2 Parameter Dynamics

The soliton parameters that were introduced in the previous subsection are now defined as

$$\kappa(z) = \frac{i}{2} D(z) \frac{\int_{-\infty}^{\infty} (q q_t^* - q^* q_t) dt}{\int_{-\infty}^{\infty} |q|^2 dt} \tag{19}$$

$$C(z) = \frac{i \int_{-\infty}^{\infty} t (q q_t^* - q^* q_t) dt}{\int_{-\infty}^{\infty} t^2 |q|^2 dt} \tag{20}$$

$$\bar{B}(z) = \frac{\int_{-\infty}^{\infty} t^2 |q|^2 dt}{\int_{-\infty}^{\infty} |q|^2 dt} \tag{21}$$

$$\bar{t}(z) = \frac{\int_{-\infty}^{\infty} t |q|^2 dt}{\int_{-\infty}^{\infty} |q|^2 dt} \tag{22}$$

where $\bar{B}$ is defined as the root-mean-square (RMS) width of the soliton. In case of DM solitons, the alternately varying dispersion as seen in (2) and (5), makes the soliton breathe periodically. Thus, the soliton width does not stay constant although the energy of the soliton does. Hence, the RMS width is used in the analytical study of DM solitons. From these definitions, one can derive the variations of the soliton frequency, chirp, RMS-width as

$$\frac{d\kappa}{dz} = 0 \tag{23}$$

$$\frac{dC}{dz} = -\frac{g(z) n^3 A^2 B^2}{2^{2n-1} {}_4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)} B\left(2n, \frac{1}{2}\right)$$
$$+ D(z) \left\{ \frac{n^5 B^2 (\kappa^2 + B^2)}{2^{2n-1}(2n+1) {}_4F_3(n, n, n, 2n; n+1, n+1, n+1; -1) B(n, \frac{1}{2})} - 12C^4 \right\} \tag{24}$$
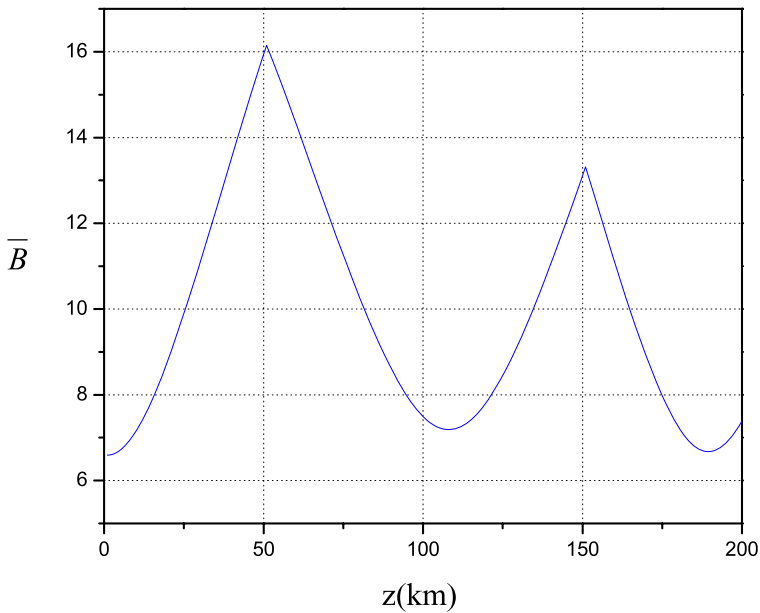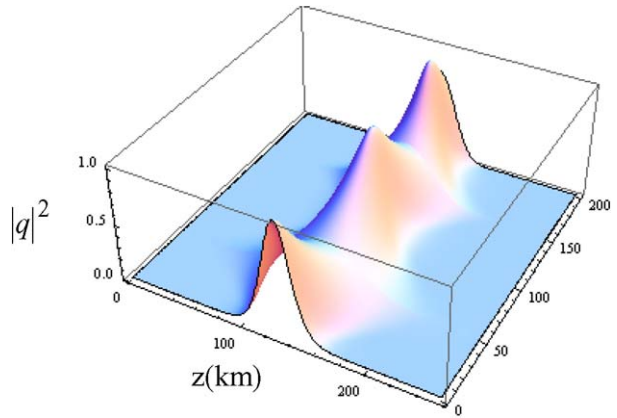
and

$$\frac{d\bar{B}}{dz} = -\frac{2^{n+1} C D(z)}{n^3 B B(n, \frac{1}{2})} {}_4F_3(n, n, n, 2n; n+1, n+1, n+1; -1) \tag{25}$$

where in (24) and (25), the generalized hypergeometric function is defined as

$$_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; x\right) = \sum_{k=0}^{n} \frac{\Pi_{j=1}^{p}(a_j)_k x^k}{\Pi_{j=1}^{q}(b_j)_k k!} \tag{26}$$

Fig. 3 Breathing super-sech
pulse for $m = 2$

Fig. 4 RMS variation of the width of the super-sech pulse

In Fig. 3, one can see the propagation of a SS pulse, for $n = 2$, through a dispersion-managed optical fiber. The corresponding RMS-width of the SS pulses is seen numerically in Fig. 4.

Also, from (22), the velocity of the soliton is given by given by

$$v = \frac{d\bar{t}}{dz} = -\kappa D(z) \tag{27}$$

4 Soliton Perturbation

In presence of perturbation terms, the perturbed NLSE is given by

$$iq_z + \frac{D(z)}{2} q_{tt} + g(z)|q|^2 q = i\epsilon R\left[q, q^*\right] \tag{28}$$

where R represents the spatio-differential operator, although sometimes it could, very well, represent an integral operator. Also, the perturbation parameter ϵ is the relative width of the spectrum and $0 < \epsilon \ll 1$ by virtue of quasi-monochromaticity [3]. In presence of these perturbation terms as in (28), the adiabatic variation of soliton energy is given by

$$\frac{dE}{dz} = \epsilon \int_{\infty}^{\infty} \left(q^* R + q R^*\right) dt \tag{29}$$

Now, from (19)–(22), one can obtain

$$\frac{d\kappa}{dz} = \frac{\epsilon D(z)}{E} \left[i \int_{-\infty}^{\infty} \left(q_t^* R - q_t R^*\right) dt - \kappa D(z) \int_{-\infty}^{\infty} \left(q^* R + q R^*\right) dx \right] \tag{30}$$

$$\frac{dC}{dz} = D(z) \left(\frac{I_{0,0,2,0,0,0,0,0}}{I_{2,2,0,0,0,0,0,0}} B^4 - 12 C^2 + \frac{I_{0,2,0,0,0,0,0,0}}{I_{2,2,0,0,0,0,0,0}} \kappa^2 B^2 \right)$$

$$- g(z) A^2 B^2 \frac{I_{0,4,0,0,0,0,0,0}}{I_{2,2,0,0,0,0,0,0}} + \frac{4\epsilon}{I_{2,2,0,0,0,0,0,0}} \frac{C B^3}{A^2} \int_{-\infty}^{\infty} t^2 \left(q^* R + q R^*\right) dx$$

$$+ \frac{i\epsilon}{I_{2,2,0,0,0,0,0,0}} \frac{B^3}{A^2} \left[2 \int_{-\infty}^{\infty} t \left(q_t^* R - q_t R^*\right) dt + \int_{-\infty}^{\infty} \left(q^* R - q R^*\right) dx \right] \tag{31}$$

$$\frac{d\bar{B}}{dz} = -4D(z) \frac{C}{B} \frac{I_{2,2,0,0,0,0,0,0}}{I_{0,2,0,0,0,0,0,0}} + \frac{\epsilon}{E} \int_{-\infty}^{\infty} t^2 \left(q^* R + q R^*\right) dx \tag{32}$$

and

$$v = \frac{d\bar{t}}{dz} = \kappa D(z) + \frac{\epsilon}{E} \int_{-\infty}^{\infty} t \left(q^* R + q R^*\right) dt \tag{33}$$

where in (31) and (32) the notation

$$I_{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8}$$

$$= \int_{-\infty}^{\infty} \tau^{p_1} f^{p_2}(\tau) \left(\frac{df}{d\tau}\right)^{p_3} \left(\frac{d^2 f}{d\tau^2}\right)^{p_4} \left(\frac{d^3 f}{d\tau^3}\right)^{p_5} \left(\frac{d^4 f}{d\tau^4}\right)^{p_6} \left(\frac{d^5 f}{d\tau^5}\right)^{p_7} \left(\frac{d^6 f}{d\tau^6}\right)^{p_8} d\tau \tag{34}$$

was used for nonnegative integers p_j for $1 \le j \le 8$.

4.1 Examples

In this paper, the following perturbation terms that are considered, are all exhaustively studied in the context of nonlinear fiber optics and optical solitons:

$$R = \delta |q|^{2p} q + \alpha q_t + \beta q_{tt} - \gamma q_{ttt} + \lambda \left(|q|^2 q\right)_t + \nu \left(|q|^2\right)_t q$$

$$+ \rho |q_t|^2 q - i\xi \left(q^2 q_t^*\right)_t - i\eta q_t^2 q^* - i\zeta q^* \left(q^2\right)_{tt} - i\mu \left(|q|^2\right)_t q$$

$$- i\chi q_{tttt} - i\psi q_{tttttt} + (\sigma_1 q + \sigma_2 q_t) \int_{-\infty}^{t} |q|^2 ds \tag{35}$$

In (35), δ is the coefficient of nonlinear damping or amplification [5] depending on its sign and p could be $0, 1, 2$. For $p = 0$, δ is the linear amplification or attenuation according to δ being positive or negative. For $p = 1$, δ represents the two-photon absorption (or a nonlinear gain if $\delta > 0$). If $p = 2$, δ gives a higher order correction (saturation or loss) to the nonlinear amplification-absorption. Also, β is the bandpass filtering term [6]. In (35), λ is the self-steepening coefficient for short pulses [5] (typically ≤ 100 femto seconds), ν is the higher order dispersion coefficient [5]. Here μ is the coefficient of Raman scattering [5, 6] and α is the frequency separation between the soliton carrier and the frequency at the peak of EDFA gain [6]. Moreover, ρ represents the coefficient of nonlinear dissipation induced by Raman scattering [7]. The coefficients of ξ, η and ζ arise due to quasi-solitons [9]. The integro-differential perturbation terms with σ_1 and σ_2 are due to saturable amplifiers [5, 6].

The coefficients of the higher order dispersion terms are respectively given by γ, χ and ψ. It is known that the NLSE, as given by (1), does not give correct prediction for pulse widths smaller than 1 picosecond. For example, in solid state solitary lasers, where pulses as short as 10 femtoseconds are generated, the approximation breaks down. Thus, quasi-monochromaticity is no longer valid and so higher order dispersion terms come in. If the group velocity dispersion is close to zero, one needs to consider the third and higher order dispersion for performance enhancement along trans-oceanic and trans-continental distances. Also, for short pulse widths where group velocity dispersion changes, within the spectral bandwidth of the signal cannot be neglected, one needs to take into account the presence of higher order dispersion terms. This reasoning leads to the inclusion of the fourth and sixth order dispersion terms that are respectively given by the coefficients of χ and ψ. For these perturbation terms, the adiabatic parameter dynamics are given as

$$
\begin{aligned}
\frac{dE}{dz} = {} & 2\epsilon A^2 \left[\frac{\delta A^{2p}}{B} B\left(np + n, \frac{1}{2}\right) \right. \\
& - \beta \left\{ \left(\frac{\kappa^2}{B} + \frac{n^2 B}{2n+1} \right) B\left(n, \frac{1}{2}\right) \right. \\
& \left. + \frac{2^{2n+1} C^2}{n^3 B^3} {}_4F_3\left(n, n, n, 2n; n+1, n+1, n+1; -1\right) \right\} \\
& - \frac{2\gamma\kappa C}{B} \left\{ B\left(n, \frac{1}{2}\right) - \frac{2^{2n}}{n(n+1)^2} \left((n+1)^2 {}_3F_2\left(n, n, 2n+1; n+1, n+1; -1\right) \right. \right. \\
& \left. \left. - n^2 {}_3F_2\left(n+1, n+1, 2n+1; n+2, n+2; -1\right) \right) \right\} \\
& + \rho A^2 \left\{ \left(\frac{\kappa^2}{B} + \frac{n^2 B}{2n+1} \right) B\left(2n, \frac{1}{2}\right) \right. \\
& \left. + \frac{2^{4n-2} C^2}{n^3 B^3} {}_4F_3\left(n, n, n, 2n; n+1, n+1, n+1; -1\right) \right\} \\
& - \frac{2\xi A^2 C}{B} \left\{ B\left(n, \frac{1}{2}\right) - \frac{2^{4n-2}}{n} {}_3F_2\left(2n, 2n, 4n+1; 2n+1, 2n+1; -1\right) \right. \\
& + \frac{n 2^{4n}}{(2n+1)^2} {}_3F_2\left(2n+1, 2n+1, 4n+1; 2n+2, 2n+2; -1\right) \\
& \left. - \frac{2^{4n-1}}{(2n+1)^2} {}_2F_1\left(2n+1, 4n+1; 2n+2; -1\right) \right\}
\end{aligned}
$$

$$- \frac{\eta A^2 C}{B} \frac{2^{4n-3}}{n(2n+1)^2} \Big\{ (2n+1)^2 \, {}_3F_2 \, (2n, 2n, 4n+1; 2n+1, 2n+1; -1)$$

$$- 4n^2 {}_3F_2 \, (2n+1, 2n+1, 4n+1; 2n+2, 2n+2; -1)$$

$$- 2n {}_2F_1 \, (2n+1, 4n+1; 2n+2; -1) \Big\}$$

$$- \frac{\zeta A^2 C}{B} \Big\{ 4B \left(2n, \frac{1}{2} \right) + \frac{5 \cdot 2^{4n-2}}{n} \, {}_3F_2 \, (2n, 2n, 4n+1; 2n+1, 2n+1; -1)$$

$$- \frac{5n2^{4n}}{(2n+1)^2} \, {}_3F_2 \, (2n+1, 2n+1, 4n+1; 2n+2, 2n+2; -1)$$

$$- \frac{5 \cdot 2^{4n-1}}{(2n+1)^2} \, {}_2F_1 \, (2n+1, 4n+1; 2n+2; -1) \Big\}$$

$$+ 4\chi C \Big\{ \frac{2n - 3n^2 B + 1}{2n+1} B \left(n, \frac{1}{2} \right)$$

$$- nB2^{2n} \, {}_3F_2 \, (n, n, 2n+1; n+1, n+1; -1)$$

$$- \frac{n^3 B 2^{2n}}{(n+1)^2} \, {}_3F_2 \, (n+1, n+1, 2n+1; n+2, n+2; -1)$$

$$+ \frac{n(n+2)B2^{2n+2}}{n+1} \, {}_3F_2 \, (n+1, n+1, 2n+3; n+2, n+2; -1)$$

$$+ \frac{n(n+1)B2^{2n+2}}{n+2} \, {}_3F_2 \, (n+2, n+2, 2n+3; n+3, n+3; -1)$$

$$+ \frac{3\kappa^2 2^{2n}}{nB} \, {}_3F_2 \, (n, n, 2n+1; n+1, n+1; -1)$$

$$- \frac{3\kappa^2 n 2^{2n}}{(n+1)^2 B} \, {}_3F_2 \, (n+1, n+1, 2n+1; n+2, n+2; -1)$$

$$+ \frac{3C^2 2^{2n+1}}{n^3 B^3} \, {}_5F_4 \, (n, n, n, n, 2n+1; n+1, n+1, n+1, n+1; -1)$$

$$- \frac{3C^2 n 2^{2n+1}}{B^3(n+1)^4} \, {}_5F_4 \, (n, n, n, n, 2n+1; n+1, n+1, n+1, n+1; -1)$$

$$- \frac{3C^2 2^{2n+1}}{B^3 n^3} \, {}_4F_3 \, (n, n, n, 2n; n+1, n+1, n+1; -1) \Big\}$$

$$+ 2\psi C \Big\{ \frac{90B^3 n^3 (4n^5 + 28n^4 + 50n^3 + 9n^2 - 40n - 24)}{(2n+1)(2n+3)} B \left(n, \frac{1}{2} \right)$$

$$- \frac{216B^3 n^3 (2n^3 + 7n^2 + 7n + 2)}{(2n+1)(2n+3)} B \left(n, \frac{1}{2} \right)$$

$$+ \frac{189n^3 \kappa^2 B}{2n+1} B \left(n, \frac{1}{2} \right) + \frac{15}{B} \left(\kappa^4 - 4C^2 \right) B \left(n, \frac{1}{2} \right)$$

$$- 3n^2 B^3 2^{2n} \, {}_3F_2 \, (n, n, 2n+1; n+1, n+1; -1)$$

$$+ \frac{3n^4 B^3 2^{2n}}{(n+1)^2} \, _3F_2 \left(n+1, n+1, 2n+1; n+2, n+2; -1\right)$$

$$+ \frac{3n B^3 2^{2n+3}(n+2)(n^2+2n+2)}{n+1} \, _3F_2 \left(n+1, n+1, 2n+3; n+2, n+2; -1\right)$$

$$- \frac{3n B^3 2^{2n+3}(n+1)(n^2+2n+2)}{n+2} \, _3F_2 \left(n+2, n+2, 2n+3; n+3, n+3; -1\right)$$

$$- \frac{6n B^3 (n+1)(n+2)(n+3)(n+4)}{(n+2)^2} \, _3F_2 \left(n+2, n+2, 2n+5; n+4, n+4; -1\right)$$

$$+ \frac{6n B^3 (n+1)(n+2)(n+3)(n+4)}{(n+3)^2} \, _3F_2 \left(n+3, n+3, 2n+5; n+4, n+4; -1\right)$$

$$+ 3n^3 B^3 2^{2n+2}; {}_3F_2 \left(n, n, 2n+1; n+1, n+1; -1\right)$$

$$- \frac{3n^4 B^3 2^{2n+2}}{(n+1)^2} \, _3F_2 \left(n+1, n+1, 2n+1; n+2, n+2; -1\right)$$

$$- 3n^3 B^3 2^{2n+5}; {}_3F_2 \left(n+1, n+1, 2n+3; n+2, n+2; -1\right)$$

$$+ \frac{3n^3 B^3 2^{2n+5}(n+1)^2}{(n+2)^2} \, _3F_2 \left(n+2, n+2, 2n+3; n+3, n+3; -1\right)$$

$$+ \frac{3n^2 B^3 2^{2n+6}(n+1)^2}{n+2} \, _3F_2 \left(n+2, n+2, 2n+5; n+3, n+3; -1\right)$$

$$- \frac{3n^2 B^3 2^{2n+5}(n+1)^2(n+2)}{(n+3)^2} \, _3F_2 \left(n+3, n+3, 2n+5; n+4, n+4; -1\right)$$

$$+ 33n\kappa^2 B 2^{2n-1} \, _3F_2 \left(n, n, 2n+1; n+1, n+1; -1\right)$$

$$- \frac{33n^3 \kappa^2 B 2^{2n-1}}{(n+1)^2} \, _3F_2 \left(n+1, n+1, 2n+1; n+2, n+2; -1\right)$$

$$- \frac{3n\kappa^2 B 2^{2n+1}(n+1)}{n+1} \, _3F_2 \left(n+2, n+2, 2n+3; n+3, n+3; -1\right)$$

$$+ \frac{3 \cdot 2^{2n}(10n^4 + 44n^3 + 66n^2 + 44n + 11)}{Bn(n+1)^4}$$

$$\times \, _5F_4 \left(n+1, n+1, n+1, n+1, 2n+1; n+2, n+2, n+2, n+2; -1\right)$$

$$+ \frac{33n(n+2)2^{2n+2}}{(n+1)^3}$$

$$\times \, _5F_4 \left(n+1, n+1, n+1, n+1, 2n+3; n+2, n+2, n+2, n+2; -1\right)$$

$$+ \frac{33n(n+2)2^{2n+2}}{B(n+2)^3}$$

$$\times \, _5F_4 \left(n+2, n+2, n+2, n+2, 2n+3; n+3, n+3, n+3, n+3; -1\right)$$

$$- \frac{3n 2^{2n+1}}{B} \, _3F_2 \left(n, n, 2n+1; n+1, n+1; -1\right)$$

$$+ \frac{3n^3 2^{2n+1}}{B(n+1)^2} \, _3F_2 \left(n+1, n+1, 2n+1; n+2, n+2; -1 \right)$$

$$+ \frac{9n^2 2^{2n+3}}{B(n+1)} \, _3F_2 \left(n+1, n+1, 2n+3; n+2, n+2; -1 \right)$$

$$+ \frac{9n^2 (n+1) 2^{2n+3}}{B (n+2)^2} \, _3F_2 \left(n+2, n+2, 2n+3; n+3, n+3; -1 \right)$$

$$- \frac{63 \cdot 2^{2n+1}}{Bn} \, _4F_3 \left(n, n, n, 2n; n+1, n+1, n+1; -1 \right)$$

$$- \frac{63n 2^{2n+3}}{B (n+1)^2} \, _4F_3 \left(n+1, n+1, n+1, 2n+2; n+2, n+2, n+2; -1 \right)$$

$$- \frac{15(\kappa^4 - 12C^2) 2^{2n}}{Bn} \, _3F_2 \left(n, n, 2n+1; n+1, n+1; -1 \right)$$

$$+ \frac{15n(\kappa^4 - 12C^2) 2^{2n}}{B (n+1)^2} \, _3F_2 \left(n+1, n+1, 2n+1; n+2, n+2; -1 \right) \Bigg\}$$

$$- \frac{\sigma_2 A^2}{B} \int_{-\infty}^{\infty} \frac{\tanh \tau}{\cosh^{2n} \tau} \left(\int_{-\infty}^{\tau} \frac{d\tau_1}{\cosh \tau_1} \right) d\tau \Bigg] \tag{36}$$

$$\frac{d\kappa}{dz} = - \frac{\epsilon D(z)}{B^4 B(n, \frac{1}{2})} \Bigg[2\beta\kappa \left\{ \frac{n^2 B^6}{2n+1} B \left(n, \frac{1}{2} \right) \right.$$

$$+ \frac{2^{2n+1} B^2 C^3}{n^3} \, _4F_3 (n, n, n, 2n; n+1, n+1, n+1; -1) \Bigg\}$$

$$\times 2\gamma C \left\{ nB^5 \left(\frac{3n}{2n+1} B \left(n, \frac{1}{2} \right) - 2^{2n} \, _3F_2 (n, n, 2n+1; n+1, n+1; -1) \right. \right.$$

$$+ \frac{n^2}{(n+1)^2} 2^{2n} \, _3F_2 (n+1, n+1, 2n+1; n+1, n+1; -1)$$

$$+ \frac{n-1}{n+1} 2^{2n+2} \, _3F_2 (n+1, n+1, 2n+3; n+2, n+2; -1)$$

$$+ \frac{n^2 - 1}{(n+2)^2} 2^{2n+2} \, _3F_2 (n+2, n+2, 2n+3; n+3, n+3; -1) \Bigg)$$

$$+ 2\kappa^2 B^4 \left(B \left(n, \frac{1}{2} \right) - \frac{2^{2n}}{n} \, _3F_2 (n, n, 2n+1; n+1, n+1; -1) \right.$$

$$+ \frac{n 2^{2n}}{(n+1)^2} \, _3F_2 (n+1, n+1, 2n+1; n+1, n+1; -1)$$

$$+ \frac{3C^2 2^{2n+1}}{n^3 B^3} \, _4F_3 (n, n, n, 2n; n+1, n+1, n+1; -1)$$

$$- \frac{3C^2 2^{2n+1}}{n^3 B^3} \, _5F_4 (n, n, n, n, 2n+1; n+1, n+1, n+1, n+1; -1)$$

$$- \frac{3n C^2 2^{2n+1}}{B^3 (n+1)^4}$$

$$\times \,_5F_4(n+1,n+1,n+1,n+1,2n+1;n+2,n+2,n+2,n+2;-1)\bigg)\bigg\}$$

$$-\frac{2^{4n-1}\rho\kappa A^2 B^2 C^2}{n^3}\,_4F_3(2n,2n,2n,4n;2n+1,2n+1,2n+1;-1)$$

$$+\frac{2^{4n}A^2 B^4 C}{(2n+1)^2}(\lambda+\nu)\,_2F_1(2n+1,4n+1;2n+2;-1)$$

$$+\frac{2^{4n+1}nA^2 B^4 C}{(2n+1)^2}(\lambda+\nu)\,_3F_2(n+1,n+1,2n+1;n+1,n+1;-1)$$

$$-\frac{2^{4n}A^2 B^4 C}{n(2n+1)^2}(\lambda+\nu)\,_3F_2(2n,2n,4n+1;2n+1,2n+1;-1)$$

$$-4\xi\kappa A^2 B^4 C B\left(2n,\frac{1}{2}\right)$$

$$-\frac{2^{4n}\kappa A^2 B^4 C}{(2n+1)^2}(8\xi-\eta+6\zeta)\,_2F_1(2n+1,4n+1;2n+2;-1)$$

$$+\frac{2^{4n+1}n\kappa A^2 B^4 C}{(2n+1)^2}(8\xi-\eta+6\zeta)\,_3F_2(2n+1,2n+1,4n+1;2n+2,2n+2;-1)$$

$$+\frac{2^{8n-7}\kappa A^2 B^4 C}{n(2n+1)^2}(8\xi-\eta+6\zeta)\,_3F_2(2n,2n,4n+1;2n+1,2n+1;-1)$$

$$+4\chi\kappa C\bigg\{2nB^6\bigg(\frac{3n}{2n+1}B\left(n,\frac{1}{2}\right)-2^{2n}\,_3F_2(n,n,2n+1;n+1,n+1;-1)$$

$$-\frac{2^{2n}n^2}{(n+1)^2}\,_3F_2(n+1,n+1,2n+1;n+2,n+2;-1)$$

$$+\frac{2^{2n+2}(n-1)}{n+1}\,_3F_2(n+1,n+1,2n+3;n+2,n+2;-1)$$

$$-\frac{2^{n+2}(n+1)}{(n+2)^2}\,_3F_2(n+2,n+2,2n+3;n+3,n+3;-1)\bigg)$$

$$-B^4\bigg(B\left(n,\frac{1}{2}\right)-\frac{2^{2n}}{n}\,_3F_2(n,n,2n+1;n+1,n+1;-1)$$

$$+\frac{n2^{2n}}{(n+1)^2}\,_3F_2(n+1,n+1,2n+1;n+2,n+2;-1)\bigg)$$

$$-\frac{3B^2 C^2 2^{2n+2}}{n^3}\,_4F_3(n,n,n,2n;n+1,n+1,n+1;-1)$$

$$+3B^2 C^2 2^{2n+2}\,_5F_4(n,n,n,n,2n+1;n+1,n+1,n+1,n+1;-1)$$

$$-\frac{3nB^2 C^2 2^{2n+2}}{(n+1)^4}$$

$$\times \,_5F_4(n+1,n+1,n+1,n+1,2n+1;n+2,n+2,n+2,n+2;-1)\bigg\}$$

$$+ 2\psi\kappa C \left\{ 2nB^8 \left(\frac{n^2(4n^5 + 28n^4 + 262n^3 + 503n^2 + 361n + 64)}{(2n+1)(2n+3)} B\left(n, \frac{1}{2}\right) \right. \right.$$

$$+ 21n^2 2^{2n} B^8 \, {}_3F_2(n, n, 2n+1; n+1, n+1; -1)$$

$$- \frac{21n^4 2^{2n}}{(n+1)^2} \, {}_3F_2(n+1, n+1, 2n+1; n+2, n+2; -1)$$

$$- \frac{2^{2n+3}(21n^3 + 50n^2 + 58n + 12)}{n+1} \, {}_3F_2(n+1, n+1, 2n+3; n+2, n+2; -1)$$

$$+ \frac{2^{2n+3}(21n^4 + 71n^3 + 108n^2 70n + 12)}{(n+2)^2}$$

$$\times \, {}_3F_2(n+2, n+2, 2n+3; n+3, n+3; -1)$$

$$+ \frac{2^{2n+2}(n+1)(21n^2 + 79n + 36)}{n+2} \, {}_3F_2(n+2, n+2, 2n+5; n+3, n+3; -1)$$

$$- \frac{2^{2n+2}(n+1)(n+2)(21n^2 + 79n + 36)}{(n+3)^2}$$

$$\left. \times \, {}_3F_2(n+3, n+3, 2n+5; n+4, n+4; -1) \right)$$

$$+ 2\kappa^2 B^6 \left(\frac{105n}{2n+1} B\left(n, \frac{1}{2}\right) + 193.2^{2n-2} \, {}_3F_2(n, n, 2n; n+1, n+1; -1) \right.$$

$$- \frac{193n^2 2^{2n-2}}{(n+1)^2} \, {}_3F_2(n+1, n+1, 2n+1; n+2, n+2; -1)$$

$$- \frac{2^{2n}(193n + 26)}{n+1} \, {}_3F_2(n+1, n+1, 2n+3; n+2, n+2; -1)$$

$$\left. + \frac{2^{2n}(n+1)(193n + 26)}{(n+2)^2} \, {}_3F_2(n+2, n+2, 2n+3; n+3, n+3; -1) \right)$$

$$- 9nB^4 C^2 2^{2n+1} \, {}_3F_2(n, n, 2n+1; n+1, n+1; -1)$$

$$+ \frac{9n^3 B^4 C^2 2^{4n+1}}{(n+1)^2} \, {}_3F_2(n+1, n+1, 2n+1; n+2, n+2; -1)$$

$$+ \frac{72n^2 B^4 C^2 2^{2n}}{n+1} \, {}_3F_2(n+1, n+1, 2n+3; n+2, n+2; -1)$$

$$- \frac{72n^2(n+1)B^4 C^2 2^{2n}}{(n+2)^2} \, {}_3F_2(n+2, n+2, 2n+3; n+3, n+3; -1)$$

$$- \frac{4862^{2n} B^4 C^2}{n} \, {}_4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)$$

$$- \frac{72n(27n + 17)2^{2n} B^4 C^2}{(n+1)^3} \, {}_4F_3(n+1, n+1, n+1, 2n+2; n+2, n+2, n+2; -1)$$

$$- \frac{513n2^{2n} C^2}{nB} \, {}_5F_4(n, n, n, n, 2n+1; n+1, n+1, n+1, n+1; -1)$$

$$- \frac{9n^3 2^{2n} C^2}{B(n+1)^4} \, {}_5F_4(n+1, n+1, n+1, n+1, 2n+1; n+2, n+2, n+2, n+2; -1)$$

$$+ \frac{15B^4 2^{2n}}{n} \left(\kappa^4 - C^2 \right) {}_3F_2(n, n, 2n+1; n+1, n+1; -1)$$

$$+ \frac{15B^4 n^3 2^{2n}}{(n+1)^4} \left(\kappa^4 - C^2 \right) {}_3F_2(n+1, n+1, 2n+1; n+2, n+2; -1)$$

$$- \frac{15\kappa^2 B^2 C^2 2^{2n+3}}{n^3} \, {}_4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)$$

$$- 15\kappa^2 B^2 C^2 2^{2n+3} \, {}_5F_3(n, n, n, n, 2n+1; n+1, n+1, n+1, n+1; -1)$$

$$- \frac{15n\kappa^2 B^2 C^2 2^{2n+3}}{(n+1)^4}$$

$$\times \, {}_5F_4(n+1, n+1, n+1, n+1, 2n+1; n+2, n+2, n+2, n+2; -1)$$

$$- \frac{9(5n+4)2^{4n+5}}{n^6} \, {}_6F_5(n, n, n, n, n, 2n; n+1, n+1, n+1, n+1, n+1; -1) \Big\}$$

$$- \sigma_2 n\kappa A^2 B^4 \int_{-\infty}^{\infty} \frac{\tanh \tau}{\cosh^{2n} \tau} \left(\int_{-\infty}^{\tau} \frac{d\tau_1}{\cosh^{2n} \tau_1} \right) d\tau \Bigg] \tag{37}$$

$$\frac{dC}{dz} = - \frac{g(z)n^3 A^2 B^2}{2^{2n-1} \, {}_4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)} B\left(2n, \frac{1}{2}\right)$$

$$+ D(z) \left\{ \frac{n^5 B^2(\kappa^2 + B^2)}{2^{2n-1}(2n+1) \, {}_4F_3(n, n, n, 2n; n+1, n+1, n+1; -1) B(n, \frac{1}{2})} - 12C^4 \right\}$$

$$+ \frac{\epsilon n^3 B^3}{2^{2n-2} \, {}_4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)} \left[\frac{nA^2 B}{4n+1} (\xi + \eta - 4\zeta) B\left(2n, \frac{1}{2}\right) \right.$$

$$+ \frac{2^{2n(p+1)+2} \delta A^{2p} C}{n^3(p+1)^3 B^3} \, {}_4F_3(pn+n, pn+n, pn+n, 2pn+2n;$$
$$pn+n+1, pn+n+1, pn+n+1; -1)$$

$$+ \frac{2^{4n-1} n^2 AB}{(2n+1)^2} (2\xi + \eta + 5\zeta) \, {}_2F_1(2n+1, 4n+1; 2n+2; -1)$$

$$- 2^{4n-2} nAB(2\xi + \eta + 5\zeta) \, {}_3F_2(n, n, 2n+1; n+1, n+1; -1)$$

$$- \frac{2^{4n} n^3 AB}{(2n+1)^2} (2\xi + \eta + 5\zeta) \, {}_3F_2(2n+1, 2n+1, 4n+1; 2n+2, 2n+2; -1)$$

$$- \frac{2^{4n} n^2 AB}{(n+1)^2} (3n\xi + \xi + n\eta + 4n\zeta + 2\zeta)$$

$$\times \, {}_3F_2(2n+2, 2n+2, 4n+3; 2n+3, 2n+3; -1)$$

$$- \frac{2^{4n+2} n^2 AB}{(2n+1)^2} (3n\xi + \xi + n\eta + 4n\zeta + 2\zeta)$$

$$\times \, {}_3F_2(2n+1, 2n+1, 4n+3; 2n+2, 2n+2; -1)$$

$$+ \frac{\kappa}{B} (\lambda + 3\kappa\xi - \kappa\eta) B\left(2n, \frac{1}{2}\right)$$

$$+ \frac{2^{4n-2}\kappa}{(2n+1)^2 B} (2\xi\kappa + \eta\kappa - 4\nu - 4\lambda) \,_2F_1(2n+1, 4n+1; 2n+2; -1)$$

$$+ \frac{2^{4n-3}\kappa}{nB} (2\xi\kappa + \eta\kappa - 4\nu - 4\lambda) \,_3F_2(n, n, 2n+1; n+1, n+1; -1)$$

$$+ \frac{2^{4n-1}n\kappa}{(2n+1)^2} (2\xi\kappa + \eta\kappa - 4\nu - 4\lambda)$$

$$\times \,_3F_2(2n+1, 2n+1, 4n+1; 2n+2, 2n+2; -1)$$

$$+ \frac{2^{4n-1}\rho C}{n^2 B} \,_4F_3(2n, 2n, 2n, 4n; 2n+1, 2n+1, 2n+1; -1)$$

$$+ \frac{2^{4n+3}\rho nC}{(2n+1)^3 B} \,_4F_3(2n+1, 2n+1, 2n+1, 4n+2; 2n+2, 2n+2, 2n+2; -1)$$

$$+ \frac{2^{4n-4}C}{n^3 B^3} \left(28\rho\kappa^2 - \eta C + 6\zeta C\right) \,_4F_3(2n, 2n, 2n, 4n; 2n+1, 2n+1, 2n+1; -1)$$

$$+ \frac{2^{4n-4}.3C^2}{n^3 B^3} (2\xi - 2\eta - 3\zeta)$$

$$\times \,_5F_4(2n, 2n, 2n, 2n, 4n+1; 2n+1, 2n+1, 2n+1, 2n+1; -1)$$

$$- \frac{2^{4n}.3C^2}{n^3(2n+1)^4 B^3} (2\xi - 2\eta - 3\zeta)$$

$$\times \,_5F_4(2n+1, 2n+1, 2n+1, 2n+1, 4n+1; 2n+2, 2n+2, 2n+2, 2n+2; -1)$$

$$+ \frac{2^{4n-4}.9\rho C^3}{n^5 B^5}$$

$$\times \,_6F_5(2n, 2n, 2n, 2n, 2n, 4n; 2n+1, 2n+1, 2n+1, 2n+1, 2n+1; -1)$$

$$+ \frac{3n^2\gamma\kappa B}{2n+1} B\left(n, \frac{1}{2}\right) + 2^{2n+1}n\gamma\kappa B \,_3F_2(n, n, 2n+1; n+1, n+1; -1)$$

$$- \frac{2^{2n+1}n^3\gamma\kappa B}{(n+1)^2} \,_3F_2(n+1, n+1, 2n+1; n+2, n+2; -1)$$

$$- \frac{2^{2n+3}n(n-1)\gamma\kappa B}{n+1} \,_3F_2(n+1, n+1, 2n+3; n+2, n+2; -1)$$

$$- \frac{2^{2n+3}n(n-1)(n+1)\gamma\kappa B}{(n+2)^2} \,_3F_2(n+2, n+2, 2n+3; n+3, n+3; -1)$$

$$+ \frac{1}{B} \left(\alpha\kappa - 2\beta C + \gamma\kappa^3 + 6\gamma\kappa^2 C - 4\gamma C^2\right) B\left(n, \frac{1}{2}\right)$$

$$+ \frac{2^{2n+1}}{nB} \left(2\beta C - \kappa^3\gamma\right) \,_3F_2(n, n, 2n+1; n+1, n+1; -1)$$

$$- \frac{2^{2n+1}n}{(n+1)^2 B} \left(2\beta C - \kappa^3\gamma\right) \,_3F_2(n+1, n+1, 2n+1; n+2, n+2; -1)$$

$$+ \frac{2^{2n+4}\beta C}{(n+1)^3 B} \,_4F_3(n+1, n+1, n+1, 2n+2; n+2, n+2, n+2; -1)$$

$$-\frac{2^{2n+1}C}{n^3B^3}(3\beta\kappa+\beta\kappa C+10\gamma C)\ {}_4F_3(n,n,n,2n;n+1,n+1,n+1;-1)$$

$$+\frac{2^{2n+1}.21\beta\kappa C^2}{n^3B^3}\ {}_5F_4(n,n,n,n,2n+1;n+1,n+1,n+1,n+1;-1)$$

$$-\frac{2^{2n+1}.63n\beta\kappa C^2}{(n+1)^4B^3}$$

$$\times\ {}_5F_4(n+1,n+1,n+1,n+1,2n+1;n+2,n+2,n+2,n+2;-1)$$

$$-\frac{2^{4n+4}.3\beta C^3}{n^5B^5}\ {}_6F_5(n,n,n,n,n,2n;n+1,n+1,n+1,n+1,n+1;-1)$$

$$+\chi\left\{\frac{n^2(4n^5+44n^4+90n^3+43n^2-32n-24)B^3}{(2n+1)(2n+3)}B\left(n,\frac{1}{2}\right)\right.$$

$$-2^{2n}n^3B^3\ {}_3F_2(n,n,2n+1;n+1,n+1;-1)$$

$$+\frac{2^{2n}n^5B^3}{(n+1)^2}\ {}_3F_2(n+1,n+1,2n+1;n+2,n+2;-1)$$

$$+\frac{2^{2n+3}n^2(n^2+2n+2)B^3}{n+1}\ {}_3F_2(n+1,n+1,2n+3;n+2,n+2;-1)$$

$$-\frac{2^{2n+3}n^2(n+1)(n^2+2n+2)B^3}{(n+2)^2}\ {}_3F_2(n+2,n+2,2n+3;n+3,n+3;-1)$$

$$-\frac{2^{2n+4}n^2(n+1)(n+3)B^3}{n+2}\ {}_3F_2(2n+2,2n+2,4n+3;2n+3,2n+3;-1)$$

$$+\frac{2^{2n+4}n^2(n+1)(n+2)B^3}{n+3}\ {}_3F_2(n+3,n+3,2n+5;n+4,n+4;-1)$$

$$+\frac{6n^2\kappa^2B}{2n+1}B\left(n,\frac{1}{2}\right)+2^{2n+1}n\kappa^2B\ {}_3F_2(n,n,2n+1;n+1,n+1;-1)$$

$$-\frac{2^{2n+1}n^3B}{(n+1)^2}\ {}_3F_2(n+1,n+1,2n+1;n+2,n+2;-1)$$

$$+\frac{8n(n+4)\kappa^2B}{n+1}\ {}_3F_2(n+1,n+1,2n+3;n+2,n+2;-1)$$

$$-\frac{32n(n+1)(n+4)\kappa^2B}{(n+2)^2}\ {}_3F_2(n+2,n+2,2n+3;n+3,n+3;-1)$$

$$+\frac{\kappa^4-12C^2}{B}B\left(n,\frac{1}{2}\right)+\frac{2^{2n}.3}{nB}\left(\kappa^4+12C^2\right)\ {}_3F_2(n,n,2n+1;n+1,n+1;-1)$$

$$+\frac{2^{2n}.3n}{(n+1)^2}\left(\kappa^4+12C^2\right)\ {}_3F_2(n+1,n+1,2n+1;n+2,n+2;-1)$$

$$+\frac{2^{2n+2}.21C^2}{n^2B}\ {}_4F_3(n,n,n,2n;n+1,n+1,n+1;-1)$$

$$-\frac{2^{2n+4}.3C^2(7n+3)}{(n+1)^3B}\ {}_4F_3(n+1,n+1,n+1,2n+2;n+2,n+2,n+2;-1)$$

$$+ \frac{2^{2n+2}.3C^2}{n^2 B} {}_5F_4(n,n,n,n,2n+1;n+1,n+1,n+1,n+1;-1)$$

$$- \frac{2^{2n+2}.3n^2C^2}{(n+1)^4 B}$$

$$\times {}_5F_4(n+1,n+1,n+1,n+1,2n+1;n+2,n+2,n+2,n+2;-1)$$

$$- \frac{2^{2n+2}.3(n+8)C^2}{(n+1)^3 B}$$

$$\times {}_5F_4(n+1,n+1,n+1,n+1,2n+3;n+2,n+2,n+2,n+2;-1)$$

$$+ \frac{2^{2n+4}.3(n+1)(n+8)}{(n+2)^4 B}$$

$$\times {}_5F_4(n+2,n+2,n+2,n+2,2n+3;n+3,n+3,n+3,n+3;-1)$$

$$- \frac{2^{2n+2}.3\kappa^2C^2}{n^3 B^3} {}_4F_3(n,n,n,2n;n+1,n+1,n+1;-1)$$

$$- \frac{2^{2n+2}.45\kappa^2C^2}{n^3 B^3} {}_5F_4(n,n,n,n,2n+1;n+1,n+1,n+1,n+1;-1)$$

$$- \frac{2^{2n+2}.3nC^2}{(n+1)^4 B^3}$$

$$\times {}_5F_4(n+1,n+1,n+1,n+1,2n+1;n+2,n+2,n+2,n+2;-1)$$

$$- \frac{2^{4n+3}.3(23n-16)C^4}{n^6 B^5}$$

$$\times {}_6F_5(n,n,n,n,n,2n;n+1,n+1,n+1,n+1,n+1;-1) \Bigg\}$$

$$+ \psi \Bigg\{ \Big\{ \big(n(32n^8 + 335n^7 + 1706n^6 + 4951n^5 + 7676n^4 + 4296n^3 - 3508n^2$$

$$- 5840n - 2160) \big) \big/ \big((2n+1)(2n+3)(2n+5) \big) \Big\}$$

$$\times B\left(n, \frac{1}{2} \right) - 2^{2n}n^5 B^5 {}_3F_2(n,n,2n+1;n+1,n+1;-1)$$

$$+ \frac{2^{2n}n^9 B^5}{(n+1)^2} {}_3F_2(n+1,n+1,2n+1;n+2,n+2;-1)$$

$$+ \frac{2^{2n+2}n(n^4 + 3n^3 + 7n^2 + 8n + 4)B^5}{n+1}$$

$$\times {}_3F_2(n+1,n+1,2n+3;n+2,n+2;-1)$$

$$- \frac{2^{2n+2}n(n+1)(n^4 + 3n^3 + 7n^2 + 8n + 4)B^5}{(n+2)^2}$$

$$\times {}_3F_2(n+2,n+2,2n+3;n+3,n+3;-1)$$

$$+ 2^{2n+3}.3n^2(n+1)\left(n^2 + 5n + 10 \right) B^5 {}_3F_2(n+2,n+2,2n+5;n+3,n+3;-1)$$

$$- \frac{2^{2n+3}.3n^2(n+1)(n+2)^2(n^2 + 5n + 10)B^5}{(n+3)^2}$$

$$\times \ _3F_2(n+3, n+3, 2n+5; n+4, n+4; -1)$$

$$+ \frac{2^{2n+5} n^2 (n+1)(n+2)(n+4)(n+5) B^5}{n+3}$$

$$\times \ _3F_2(n+3, n+3, 2n+7; n+4, n+4; -1)$$

$$- \frac{2^{2n+5} n^2 (n+1)(n+2)(n+3)(n+5) B^5}{n+4}$$

$$\times \ _3F_2(n+4, n+4, 2n+7; n+5, n+5; -1)$$

$$- \frac{2^{2n}.15 n^2 (4n^5 + 28n^2 + 50n^3 + 9n^2 - 40n - 24)\kappa^2 B^3}{(2n+1)(2n+3)} B\left(n, \frac{1}{2}\right)$$

$$+ 2^{2n}.9 n^2 \kappa^2 \ _3F_2(n, n, 2n+1; n+1, n+1; -1)$$

$$+ 2^{2n}.9 n^2 \kappa^2 \ _3F_2(n+1, n+1, 2n+1; n+2, n+2; -1)$$

$$- \frac{2^{2n+3}.3(3n^3 - 4n^2 - 30n - 40)\kappa^2}{n+1} \ _3F_2(n+1, n+1, 2n+3; n+2, n+2; -1)$$

$$- \frac{2^{2n+3}.3(n+1)(3n^3 - 4n^2 - 30n - 40)\kappa^2}{(n+2)^2}$$

$$\times \ _3F_2(n+2, n+2, 2n+3; n+3, n+3; -1)$$

$$+ \frac{2^{2n+4}.3(n+1)(n+3)(3n-8)\kappa^2}{n+2} \ _3F_2(n+2, n+2, 2n+5; n+3, n+3; -1)$$

$$- \frac{2^{2n+4}.3(n+1)(n+2)}{n+3} \ _3F_2(n+3, n+3, 2n+5; n+4, n+4; -1)$$

$$- \frac{n^2 B}{2n+1}\left(\kappa^4 + 12C^2\right) B\left(n, \frac{1}{2}\right) - 2^{2n+1}\left(5\kappa^4 - 16C^2\right)$$

$$\times \ _3F_2(n, n, 2n+1; n+1, n+1; -1)$$

$$- \frac{2^{2n-1} n^2}{(n+1)^2} \ _3F_2(n+1, n+1, 2n+1; n+2, n+2; -1)$$

$$- \frac{8nB(32nC^2 - 30C^2 - 10n\kappa^4 - 5\kappa^4)}{n+1}$$

$$\times \ _3F_2(n+1, n+1, 2n+3; n+2, n+2; -1)$$

$$- \frac{8Bn(n+1)(32nC^2 - 30C^2 - 10n\kappa^4 - 5\kappa^4)}{(n+2)^2}$$

$$\times \ _3F_2(n+2, n+2, 2n+3; n+3, n+3; -1)$$

$$+ 263nBC^2 \ _4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)$$

$$- \frac{16C^2(67n^2 + 45n + 45)}{(n+1)^2}$$

$$\times \ _4F_3(n+1, n+1, n+1, 2n+2; n+2, n+2, n+2; -1)$$

$$- \frac{80nC^2(2n^2 + 3n + 2)}{(n+1)^3} \ _4F_3(n+1, n+1, n+1, 2n+2; n+2, n+2, n+2; -1)$$

$$+ 9C^2 \, _5F_4(n, n, n, n, 2n+1; n+1, n+1, n+1, n+1; -1)$$

$$- \frac{9n^4C^2}{(n+1)^4} \, _5F_4(n+1, n+1, n+1, n+1, 2n+1; n+2, n+2, n+2, n+2; -1)$$

$$+ \frac{36(n+2)(3n^2+6n+16)}{(n+1)^3}$$

$$\times \, _5F_4(n+1, n+1, n+1, n+1, 2n+3; n+2, n+2, n+2, n+2; -1)$$

$$- \frac{36(n+1)(n+2)(3n^2+6n+16)}{(n+2)^4}$$

$$\times \, _5F_4(n+2, n+2, n+2, n+2, 2n+3; n+3, n+3, n+3, n+3; -1)$$

$$+ \frac{16(136n^3+319n^2+102n-81)}{(n+2)^3}$$

$$\times \, _4F_3(n+2, n+2, n+2, 2n+4; n+3, n+3, n+3; -1)$$

$$- \frac{\kappa^2}{B} \left(\kappa^4 - 36C^2\right) B\left(n, \frac{1}{2}\right) - \frac{2^{2n}\kappa^2}{nB} \left(5\kappa^4 - 252C^2\right)$$

$$\times \, _3F_2(n, n, 2n+1; n+1, n+1; -1)$$

$$+ \frac{2^{2n}n\kappa^2}{(n+1)^2B} \left(5\kappa^4 - 252C^2\right) \, _3F_2(n+1, n+1, 2n+1; n+2, n+2; -1)$$

$$\times \frac{1080\kappa^2C^2}{nB} \, _4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)$$

$$\times \frac{720n(6n+7)\kappa^2C^2}{(n+1)^3B} \, _4F_3(n+1, n+1, n+1, 2n+2; n+2, n+2, n+2; -1)$$

$$+ \frac{3240\kappa^2C^2}{nB} \, _5F_4(n, n, n, n, 2n+1; n+1, n+1, n+1, n+1; -1)$$

$$- \frac{3240n^3C^2}{(n+1)^4B} \, _5F_4(n+1, n+1, n+1, n+1, 2n+1; n+2, n+2, n+2, n+2; -1)$$

$$- \frac{4320n(n-2)\kappa^2C^2}{(n+1)^3B}$$

$$\times \, _5F_4(n+1, n+1, n+1, n+1, 2n+3; n+2, n+2, n+2, n+2; -1)$$

$$- \frac{4320n(n+1)(n-2)\kappa^2C^2}{(n+2)^4B}$$

$$\times \, _5F_4(n+2, n+2, n+2, n+2, 2n+3; n+3, n+3, n+3, n+3; -1)$$

$$+ \frac{2^{2n+1}.3C^2}{n^3B^3} \left(55\kappa^4 - 92C^2\right) \, _4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)$$

$$+ \frac{2^{2n+1}.3C^3}{n^3B^3} \left(105\kappa^4 - 564C^2\right)$$

$$\times \, _5F_4(n, n, n, n, 2n+1; n+1, n+1, n+1, n+1; -1)$$

$$- \frac{2^{2n+1}.3nC^2}{(n+1)^4B^3} \left(105\kappa^4 - 564C^2\right)$$

$$\times \; _5F_4(n+1, n+1, n+1, n+1, 2n+1; n+2, n+2, n+2, n+2; -1)$$

$$-\frac{2^{4n+4}.201C^4}{n^3 B^3}\; _6F_5(n, n, n, n, n, 2n; n+1, n+1, n+1, n+1, n+1; -1)$$

$$-\frac{2^{2n+3}n(111n+1365)C^4}{(n+1)^5 B^3}\; _6F_5(n+1, n+1, n+1, n+1, n+1, 2n+2;$$
$$n+2, n+2, n+2, n+2, n+2; -1)$$

$$+\frac{2^{4n+3}.45(31n-36)\kappa^2 C^4}{n^6 B^5}$$

$$\times \; _6F_5(n, n, n, n, n, 2n; n+1, n+1, n+1, n+1, n+1; -1)$$

$$+\frac{2^{2n+3}.45(59n-77)C^6}{n^8 B^7}$$

$$\times \; _8F_7(n, n, n, n, n, n, n, 2n; n+1, n+1, n+1, n+1, n+1, n+1, n+1; -1)\Bigg\}$$

$$-\frac{2\sigma_1 C A^2}{B^3}\int_{-\infty}^{\infty}\frac{\tau}{\cosh^{2n}\tau}\left(\int_{-\infty}^{\tau}\frac{d\tau_1}{\cosh^{2n}\tau_1}\right)d\tau$$

$$-\frac{4n\sigma_2 A^2}{B^3}\int_{-\infty}^{\infty}\frac{\tau^2\tanh\tau}{\cosh\tau}\left(\int_{-\infty}^{\tau}\frac{d\tau_1}{\cosh^{2n}\tau_1}\right)d\tau\Bigg] \tag{38}$$

$$\frac{d\bar{B}}{dz}=-\frac{2^{n+1}CD(z)}{n^3 B B(n,\frac{1}{2})}\; _4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)$$

$$+\frac{2\epsilon B}{B(n,\frac{1}{2})}\Bigg[\frac{\delta A^{2p}2^{2n(p+1)-1}}{n^3(p+1)^3 B}\; _4F_3(pn+n, pn+n, pn+n, 2pn+2n;$$
$$pn+n+1, pn+n+1, pn+n+1; -1)$$

$$+\beta\Bigg\{\frac{2^{2n-1}}{nB}\; _4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)$$

$$-\frac{n2^{2n+1}}{(n+1)^2 B}\; _4F_3(n+1, n+1, n+1, 2n+2; n+2, n+2, n+2; -1)$$

$$+\frac{\kappa^2 C2^{2n-2}}{n^3 B^3}\; _4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)$$

$$-\frac{3C^2 2^{4n-2}}{n^5 B^5}\; _6F_5(n, n, n, n, n, 2n; n+1, n+1, n+1, n+1, n+1; -1)\Bigg\}$$

$$-\gamma\Bigg\{\frac{C}{B}B\left(n,\frac{1}{2}\right)+\frac{\kappa C^2 2^{2n-1}}{n^3 B^3}\; _4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)$$

$$-\frac{3\kappa C2^{2n-1}}{n^3 B^3}\; _5F_4(n, n, n, n, 2n+1; n+1, n+1, n+1, n+1; -1)$$

$$+\frac{3n\kappa C2^{2n-1}}{(n+1)^4 B^3}$$

$$\times \; _5F_4(n+1, n+1, n+1, n+1, 2n+1; n+2, n+2, n+2, n+2; -1)\Bigg\}$$

$$+ \frac{(\xi + 2\zeta)A^2 C 2^{4n-6}}{n^3 B^3} \, {}_4F_3(2n, 2n, 2n, 4n; 2n+1, 2n+1, 2n+1; -1)$$

$$+ \frac{3(\eta - \xi)A^2 C 2^{4n-6}}{n^3 B^3}$$

$$\times \, {}_5F_4(2n, 2n, 2n, 2n, 4n+1; 2n+1, 2n+1, 2n+1, 2n+1; -1)$$

$$- \frac{3(\eta - \xi)A^2 2^{4n-2}}{n^3 (2n+1)^4 B^3}$$

$$\times \, {}_5F_4(2n+1, 2n+1, 2n+1, 2n+1, 4n+1; 2n+2, 2n+2, 2n+2, 2n+2; -1) \Big]$$

$$+ \rho \Big\{ \frac{A^2 2^{4n-4}}{n^2 B} \, {}_4F_3(2n, 2n, 2n, 4n; 2n+1, 2n+1, 2n+1; -1)$$

$$- \frac{n^2 A^2 2^{4n}}{(2n+1)^3 B} \, {}_4F_3(2n+1, 2n+1, 2n+1, 4n+2; 2n+2, 2n+2, 2n+2; -1)$$

$$+ \frac{\kappa^2 A^2 2^{4n-4}}{n^3 B^3} \, {}_4F_3(2n, 2n, 2n, 4n; 2n+1, 2n+1, 2n+1; -1)$$

$$\times \frac{3A^2 C 2^{4n-5}}{n^5 B^5}$$

$$\times \, {}_6F_5(2n, 2n, 2n, 2n, 2n, 4n; 2n+1, 2n+1, 2n+1, 2n+1, 2n+1; -1) \Big\}$$

$$+ 4\chi \Big\{ \frac{3C 2^{2n-1}}{nB} \, {}_4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)$$

$$- \frac{3nC 2^{2n+1}}{(n+1)^2 B} \, {}_4F_3(n+1, n+1, n+1, 2n+2; n+2, n+2, n+2; -1)$$

$$- \frac{3C 2^{2n-1}}{nB} \, {}_5F_4(n, n, n, n, 2n+1; n+1, n+1, n+1, n+1; -1)$$

$$+ \frac{3n^3 2^{2n-1}}{(n+1)^4 B} \, {}_5F_4(n+1, n+1, n+1, n+1, 2n+1; n+2, n+2, n+2, n+2; -1)$$

$$+ \frac{3n(n+2) 2^{2n+1}}{(n+1)^3 B}$$

$$\times \, {}_5F_4(n+1, n+1, n+1, n+1, 2n+3; n+2, n+2, n+2, n+2; -1)$$

$$- \frac{3n(n+1) 2^{2n+1}}{(n+2)^3 B}$$

$$\times \, {}_5F_4(n+2, n+2, n+2, n+2, 2n+3; n+3, n+3, n+3, n+3; -1)$$

$$- \frac{3\kappa^2 C 2^{2n-1}}{n^3 B^3} \, {}_4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)$$

$$+ \frac{9\kappa^2 C 2^{2n-1}}{n^3 B^3} \, {}_5F_4(n, n, n, n, 2n+1; n+1, n+1, n+1, n+1; -1)$$

$$+ \frac{9n^4 \kappa^2 C 2^{2n-1}}{n^3 (n+1)^4 B^3}$$

$$\times {}_5F_4(n+1, n+1, n+1, n+1, 2n+1; n+2, n+2, n+2, n+2; -1)$$

$$+ \frac{3(3n+4)C^3 2^{4n+1}}{n^6 B^5} {}_6F_5(n, n, n, n, n, 2n; n+1, n+1, n+1, n+1, n+1; -1) \Bigg\}$$

$$\times 2\psi \Bigg\{ 3C 2^{2n+2} {}_4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)$$

$$- \frac{3(n^2+n+1)C 2^{2n+3}}{(n+1)^2} {}_4F_3(n+1, n+1, n+1, 2n+2; n+2, n+2, n+2; -1)$$

$$+ 9C^2 2^{2n-1} {}_5F_4(n, n, n, n, 2n+1; n+1, n+1, n+1, n+1; -1)$$

$$+ \frac{9n^4 C^2 2^{2n-1}}{(n+1)^4}$$

$$\times {}_5F_4(n+1, n+1, n+1, n+1, 2n+1; n+2, n+2, n+2, n+2; -1)$$

$$+ \frac{9C(n+2)(n^2+2n+2)2^{2n+2}}{(n+1)^3}$$

$$\times {}_5F_4(n+1, n+1, n+1, n+1, 2n+3; n+2, n+2, n+2, n+2; -1)$$

$$- \frac{9C(n+1)(n^2+2n+2)2^{2n+2}}{(n+2)^3}$$

$$\times {}_5F_4(n+2, n+2, n+2, n+2, 2n+3; n+3, n+3, n+3, n+3; -1)$$

$$- \frac{3C(n+1)(n+3)(n+5)2^{2n+3}}{(n+2)^3}$$

$$\times {}_4F_3(n+2, n+2, n+2, 2n+4; n+3, n+3, n+3; -1)$$

$$- \frac{15\kappa^2 C 2^{2n}}{n} {}_4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)$$

$$- \frac{15n C 2^{2n+2}}{(n+1)^2} {}_4F_3(n+1, n+1, n+1, 2n+2; n+2, n+2, n+2; -1)$$

$$- \frac{15\kappa^2 C 2^{2n}}{n} {}_5F_4(n, n, n, n, 2n+1; n+1, n+1, n+1, n+1; -1)$$

$$- \frac{15n^3\kappa^2 C}{(n+1)^4} {}_5F_4(n+1, n+1, n+1, n+1, 2n+1; n+2, n+2, n+2, n+2; -1)$$

$$+ \frac{n(n+2)2^{2n+2}}{(n+1)^3}$$

$$\times {}_5F_4(n+1, n+1, n+1, n+1, 2n+3; n+2, n+2, n+2, n+2; -1)$$

$$- \frac{n(n+1)2^{2n+2}}{(n+2)^3}$$

$$\times {}_5F_4(n+2, n+2, n+2, n+2, 2n+3; n+3, n+3, n+3, n+3; -1)$$

$$+ \frac{15C(\kappa^4 - 4C^2)2^{2n-1}}{n^3 B^3} {}_4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)$$

$$- \frac{45C(\kappa^4 - 12C^2)2^{2n-1}}{n^3 B^3} {}_5F_4(n, n, n, n, 2n+1; n+1, n+1, n+1, n+1; -1)$$

$$+ \frac{5n^4 C(\kappa^4 - 12C^2)2^{2n-1}}{n^3(n+1)^4 B^3}$$

$$\times {}_5F_4(n+1, n+1, n+1, n+1, 2n+1; n+2, n+2, n+2, n+2; -1)$$

$$- \frac{125C^3 2^{4n+2}}{n^3 B^3} {}_6F_5(n, n, n, n, n, 2n; n+1, n+1, n+1, n+1, n+1; -1)$$

$$- \frac{15nC^3(4n+9)2^{2n+3}}{(n+1)^5 B^3} {}_6F_5(n+1, n+1, n+1, n+1, n+1, 2n+2;$$
$$n+2, n+2, n+2, n+2, n+2; -1)$$

$$+ \frac{45\kappa^2 C^3(3n-4)2^{4n+2}}{n^6 B^5} {}_6F_5(n, n, n, n, n, 2n; n+1, n+1, n+1, n+1, n+1; -1)$$

$$+ \frac{135C^5(5n-7)2^{2n+1}}{n^8 B^7}$$

$$\times {}_8F_7(n, n, n, n, n, n, n, 2n; n+1, n+1, n+1, n+1, n+1, n+1, n+1; -1)\Big\}$$

$$- \frac{n\sigma_2 A^2}{B^3} \int_{-\infty}^{\infty} \frac{\tau^2 \tanh\tau}{\cosh^{2n}\tau} \left(\int_{-\infty}^{\tau} \frac{d\tau_1}{\cosh^{2n}\tau_1} \right) d\tau \Bigg] \tag{39}$$

while the velocity change is given by

$$v = -\kappa D(z) + \frac{2\epsilon}{B^4 B(n, \frac{1}{2})} \Bigg[\frac{2^{2n+1}\beta\kappa B^2 C}{n} {}_4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)$$

$$- \frac{2^{2n-1}\alpha}{nB} {}_3F_2(n, n, 2n+1; n+1, n+1; -1)$$

$$+ \gamma B^2 C \Bigg\{ \frac{3C 2^{2n+1}}{n} {}_4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)$$

$$- \frac{9B^3}{n^3} {}_5F_4(n, n, n, n, 2n+1; n+1, n+1, n+1, n+1; -1)$$

$$- \frac{9B^3}{(n+1)^4} {}_5F_4(n+1, n+1, n+1, n+1, 2n+1; n+2, n+2, n+2, n+2; -1) \Bigg\}$$

$$- 2^{8n-5}\rho\kappa A^2 BC \, {}_4F_3(2n, 2n, 2n, 4n; 2n+1, 2n+1, 2n+1; -1)$$

$$+ \frac{2^{4n-2}A^2 B^4}{(2n+1)^2}(3\lambda + 2v) \, {}_2F_1(2n+1, 4n+1; 2n+2; -1)$$

$$+ \frac{4\kappa A^2 B^4}{(2n+1)^2}(\xi - \eta + 4\zeta) \, {}_2F_1(2n+1, 4n+1; 2n+2; -1)$$

$$+ 4\chi \Bigg\{ 2^{2n-1}n\kappa B^6 \, {}_3F_2(n, n, 2n+1; n+1, n+1; -1)$$

$$- \frac{2^{2n-1}n^3\kappa B^6}{(n+1)^2} {}_3F_2(n+1, n+1, 2n+1; n+2, n+2; -1)$$

$$- \frac{2^{2n+1}n(n+2)\kappa B^6}{n+1} {}_3F_2(n+1, n+1, 2n+3; n+2, n+2; -1)$$

$$+ \frac{2^{2n+1} n(n+1)\kappa B^6}{n+2} \, {}_3F_2(n+2, n+2, 2n+3; n+3, n+3; -1)$$

$$- \frac{2^{2n-1} \kappa^3 B^4}{n} \, {}_3F_2(n, n, 2n+1; n+1, n+1; -1)$$

$$- \frac{2^{2n-1} n \kappa^3 B^4}{(n+1)^2} \, {}_3F_2(n+1, n+1, 2n+1; n+2, n+2; -1)$$

$$+ \frac{3\kappa B^3 C^2 2^{2n+1}}{n^3} \, {}_4F_3(n, n, n, 2n; n+1, n+1, n+1; -1)$$

$$- \frac{9\kappa B^2 C^2 2^{2n}}{n^3} \, {}_5F_4(n, n, n, n, 2n+1; n+1, n+1, n+1, n+1; -1)$$

$$+ \frac{9\kappa n B^2 C^2 2^{2n}}{(n+1)^4}$$

$$\times \, {}_5F_4(n+1, n+1, n+1, n+1, 2n+1; n+2, n+2, n+2, n+2; -1) \Big\}$$

$$+ 2\psi \left\{ \frac{15 B^8 n^2 (4n^5 + 28n^4 + 50n^3 + 9n^2 - 40n - 24)}{(2n+1)(2n+3)} B\left(n, \frac{1}{2}\right) \right.$$

$$+ 3n^3 \kappa B^8 2^{2n-1} \, {}_3F_2(n, n, 2n+1; n+1, n+1; -1)$$

$$- \frac{3n^5 \kappa B^8 2^{2n-1}}{n+1} \, {}_3F_2(n+1, n+1, 2n+1; n+2, n+2; -1)$$

$$- \frac{3n(n+2)(n^2+2n+2)\kappa B^8 2^{2n+2}}{n+1} \, {}_3F_2(n+1, n+1, 2n+3; n+2, n+2; -1)$$

$$+ \frac{3n(n+1)(n^2+2n+2)\kappa B^8 2^{2n+2}}{n+2} \, {}_3F_2(n+2, n+2, 2n+3; n+3, n+3; -1)$$

$$+ \frac{3n(n+1)(n+3)(n+4)\kappa B^8 2^{2n+3}}{n+2} \, {}_3F_2(n+2, n+2, 2n+5; n+3, n+3; -1)$$

$$- \frac{3n(n+1)(n+2)(n+4)\kappa B^8 2^{2n+3}}{n+3} \, {}_3F_2(n+3, n+3, 2n+5; n+4, n+4; -1)$$

$$- 5n\kappa^3 B^6 2^{2n} \, {}_3F_2(n, n, 2n; n+1, n+1; -1)$$

$$+ \frac{5n^3 \kappa^3 B^6 2^{2n}}{(n+1)^2} \, {}_3F_2(n+1, n+1, 2n+1; n+2, n+2; -1)$$

$$+ \frac{5n(n+2)\kappa^3 B^6 2^{2n+2}}{n+1} \, {}_3F_2(n+1, n+1, 2n+3; n+2, n+2; -1)$$

$$- \frac{5n(n+1)\kappa^3 B^6 2^{2n+2}}{n+2} \, {}_3F_2(n+2, n+2, 2n+3; n+3, n+3; -1)$$

$$+ \frac{3\kappa B^4 (\kappa^4 - 84C^2) 2^{2n-1}}{n} \, {}_3F_2(n, n, 2n+1; n+1, n+1; -1)$$

$$- \frac{3n\kappa B^4 (\kappa^4 - 84C^2) 2^{2n-1}}{n} \, {}_3F_2(n+1, n+1, 2n+1; n+2, n+2; -1)$$

$$+ \frac{324\kappa B^4 C^2}{n}\, {}_4F_3(n,n,n,2n;n+1,n+1,n+1;-1)$$

$$- \frac{144n(16n+7)\kappa B^4 C^2}{(n+1)^3}\, {}_4F_3(n+1,n+1,n+1,2n+2;n+2,n+2,n+2;-1)$$

$$- \frac{180\kappa B^4 C^2}{n}\, {}_5F_4(n,n,n,n,2n+1;n+1,n+1,n+1,n+1;-1)$$

$$+ \frac{720n(n+2)\kappa B^4 C^2}{(n+1)^3}$$

$$\times\, {}_5F_4(n+1,n+1,n+1,n+1,2n+3;n+2,n+2,n+2,n+2;-1)$$

$$- \frac{720n(n+1)B^5}{(n+2)^3}$$

$$\times\, {}_5F_4(n+2,n+2,n+2,n+2,2n+3;n+3,n+3,n+3,n+3;-1)$$

$$- \frac{15\kappa^3 B^2 C^2 2^{2n+2}}{n^3}\, {}_4F_3(n,n,n,2n;n+1,n+1,n+1;-1)$$

$$+ \frac{45\kappa^3 B^2 C^2 2^{2n+1}}{n^3}\, {}_5F_4(n,n,n,n,2n+1;n+1,n+1,n+1,n+1;-1)$$

$$- \frac{45n\kappa^3 B^2 C^2}{(n+1)^4}\, {}_5F_4(n+1,n+1,n+1,n+1,2n+1;n+2,n+2,n+2,n+2;-1)$$

$$- \frac{450(n-1)C^4 2^{4n+4}}{n^6}\, {}_3F_2(n+2,n+2,2n+3;n+3,n+3;-1)\Bigg\}$$

$$+ \frac{\sigma_1 A^2}{B^3} \int_{-\infty}^{\infty} \frac{\tau}{\cosh^{2n}\tau}\left(\int_{-\infty}^{\tau}\frac{d\tau_1}{\cosh^{2n}\tau_1}\right)d\tau\Bigg] \tag{40}$$

5 Conclusions

In this paper, the dynamics of super-sech optical solitons was studied in the context of dispersion-managed optical solitons. The adiabatic parameter dynamics of the solitons were obtained in presence of perturbation terms that are both local as well as non-local. The change in the soliton velocity is also obtained in this paper, in presence of the perturbation terms.

These results are going to be used for further study of dispersion-managed solitons. One immediate application of this is in the study of intra-channel collision of solitons by virtue of quasi-particle theory. Another application of this is in the issue of Gabitov-Turitsyn equation in presence of perturbation terms. Such applications will be ventured in future and the results of those research will be reported in future publications.

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